GAZ-460 Seat No.________

B. Sc. (Sem. VI) Examination

March / April – 2017

Mathematics : CCMATH-602

(Analysis-II)

Time : **3** Hours] [Total Marks : **70**

Instructions :
(1) All questions are compulsory.
(2) Figures to the right sude indicate marks of corresponding question.

1 (a) Let $f_1, f_2, f_3, \ldots\ldots, f_k$ be the real functions on a metric space X and let $\overline{f}$ be the mapping of X into R^k defined by
$\overline{f}(x) = \left(f_1(x), f_2(x), \ldots\ldots\ldots, f_k(x)\right), \; (x \in X)$ then $\overline{f}$ is continuous if and only if each f_i is continuous where $i = 1, 2, 3, \ldots, k$. **6**

(b) Prove that a mapping f of a metric space X into metric Y is continuous on X if and only if $f^{-1}(v)$ is open in X, for every open set v in Y. **6**

(c) If f and g are continuous mapping of metric space X into metric space Y and let E be dense subset of X. If $f(x)=g(x), \forall x \in E$ then prove that $f(x)=g(x), \forall x \in X$. 6

OR

1 (a) State and prove Taylor's theorem. 6

(b) Suppose X, Y, Z are metric spaces $E \subset X$. 6

$f: E \to Y$, $g: f(E) \to Z$ and $h: E \to Z$ defined by $h(x)=g(f(x))$, $x \in E$. If f is continuous at a point $p \in E$ and if g is continuous at a point $f(p)$ then h is continuous at p.

(c) Suppose (i) f is continuous for $x \geq 0$ (ii) $f'(x)>0$ (iii) $f(0)=0$ (iv) f' is monotonically increasing then prove that $g(x)=\dfrac{f(x)}{x}$, $x>0$ is also monotonically increasing. 6

2 (a) Suppose (1) $C_n \geq 0$ for 1,2,3,...... 6

(2) $\sum C_n$ converges (3) $\{S_n\}$ is a sequence of distinct points in (a,b) and $\alpha(x)=\sum C_n I(x-S_n)$.

If f is continuous on [a, b] then prove that

$$\int_a^b f\, da=\sum C_n f(S_n)$$

(b) Let f be a bounded real valued function defined on $[a,b]$ and α be monotonically increasing function on $[a,b]$. Prove that $f \in R(\alpha)$ on $[a,b]$ if and only if for every $\varepsilon > 0, \exists$ a partition P of $[a,b] \ni U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$. **6**

(c) Prove that $\int_0^3 x d(x - [x]) = -\frac{3}{2}$. **6**

OR

2 (a) Prove that every continuous function defines on $[a,b]$ is $R-S$ integrable on $[a,b]$. **6**

(b) Let $\in R(\alpha)$ on $[a,b]$. For $\alpha \le x \le b$, put $F(x) = \int_a^x f(t)dt$. Then prove that (i) F is continuous on $[a,b]$ (ii) If f is continuous at a point x_0 of $[a,b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$. **6**

(c) Let f be a continuous function on [0, 2] and $\alpha:[0,2]\to R$ is defined by

$$\alpha(x)=0, \quad 0\le x<1$$
$$=c, \quad x=1$$
$$=1, \quad 1<x\le 2$$

Prove that $f\in R(\alpha)$ on [0, 2].

3 (a) Suppose $f_n\in R(\alpha)$ on $[a,b]$, for $n=1,2,3,\ldots\ldots$ **6**

If $f(x)=\sum_{n=1}^{\infty} f_n(x),\ a\le x\le b$ and the series convergence uniformly on $[a,b]$ then prove that

$$\int_a^b f\,d\alpha=\sum_{n=1}^{\infty}\int_a^b f_n\,d\alpha.$$

(b) If K is a compact metric space if $f_n\in C(K), n=1,2,3,\ldots\ldots\ldots$ and if $\{f_n\}$ is pointwise bounded and equi-continuous on K then prove that $\{f_n\}$ is uniformly bounded on K. **6**

(c) Give an example of a sequence of functions for which $\lim_{n\to\infty}\left[\frac{d}{dx}f_n(x)\right]\neq\frac{d}{dx}\left[\lim_{n\to\infty}f_n(x)\right]$. **6**

OR

3 (a) Let α be monotonically increasing on $[a,b]$. Suppose $f_n \in R(\alpha)$ on $[a,b]$, for $n=1,2,3,......$ and suppose $f_n \to f$ uniformly on $[a,b]$ then prove that $f\in R(\alpha)$ on $[a,b]$ and $\int_a^b f\,d\alpha=\lim_{n\to\infty}\int_a^b f_n\,d\alpha$. **6**

(b) State and prove Cauchy criterion for uniform convergence of a sequence of functions. **6**

(c) Show that the series $\sum\frac{x^n}{n^2}$ convergence on $[0,1]$ and it can be integrating term by term. **6**

4 Attempt any two : **8**

(a) Suppose f is differentiable in (a,b). Prove

(i) If $f'(x)\geq 0$ for all $x\in(a,b)$, then f is monotonically increasing.

(ii) If $f'(x)=0$ for all $x\in(a,b)$, then f is constant.

(b) If for some partition $P=\{x_0, x_1, x_2, \ldots\ldots, x_n\}$ and $\varepsilon > 0$, $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ then prove that

(i) It also holds for its refinement.

(ii) If $S_i, t_i \in [x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots\ldots, n$ then

$$\sum_{i=1}^{n} \left| f(s_i) - f(t_i) \right| \Delta\alpha_i < \varepsilon.$$

(c) State and prove M_n – test for uniform convergence.

5 Attempt any two : 8

(a) By using definition of R-S integral evaluate

$$\int_0^1 x^2 d(3x).$$

(b) Evaluate :

(i) $\lim_{x \to 0} \frac{x - \tan x}{x^3}$

(ii) $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{\frac{1}{x}}$

(c) Discuss the convergence of following series

(i) $$\sum_{n=1}^{\infty} \frac{\sin\left(x^2 + xn^2\right)}{n(n+1)}$$

(ii) $$\cos x + \cos\frac{2x}{2^2} + \cos\frac{3x}{3^2} + \ldots\ldots$$
