GAZ-453 Seat No.________

B. Sc. (Sem. VI) Examination

March / April – 2017

Mathematics : CC - MATH - 601

(Abstract Algebra)

Time : **3** Hours] [Total Marks : **70**

Instructions : (1) This question paper contains five questions and all questions are compulsory.

(2) Figures in the **right** side indicate marks of each question.

1 (a) Define unit element and unity element in a ring. Find unit elements of $\left(Z_n;+_n;\bullet_n\right)$ if n is a composite number. **6**

(b) Suppose R be a commutative ring and $a \in R$, prove that the set $I=\{x \in R / ax=0\}$ is an ideal of ring R. **6**

(c) Show that a finite integral domain is a field. **6**

OR

1 (a) If p is a prime then show that there are exactly two elements [1] and [p–1] for which $a^{-1}=a$, in a ring $\left(z_p;+_p;\bullet_p\right)$. **6**

(b) Prove that a non-zero element [m] of a ring $\left(z_n; +_n; \bullet_n\right)$ is a zero divisor if and only if m and n are not relatively prime. **6**

(c) Give an example of a ring without unity whose subring has a unity element. **6**

2 (a) State and prove unique factorization theorem for polynomials. **6**

(b) Suppose $f(x) \in F(x)$ and $a \in F$. Then prove that the remainder on dividing $f(x)$ by $x-a$ will be $f(a)$. **6**

(c) Obtain four polynomials of degree 3 over the field Q of rational numbers out of which **6**

(i) One has only one zero in Q

(ii) One has only two zeros in Q

(iii) One has only three zeros in Q

(iv) One has no zero in Q.

OR

2 (a) Define degree of a polynomial. In usual notations, prove that $[f \cdot g] = [f] + [g]$ for non-zero polynomials f and g. **6**

(b) State and prove Gauss's Lemma. **6**

(c) Show that division algorithm is not true in $Z[x]$. **6**

3 (a) Let $(R;+;\bullet)$ be a ring with unity. Prove that the mapping $\phi:(Z,+,\bullet)\rightarrow(R,+,\bullet)$ defined by $\phi(n)=n\cdot 1,\ n\in Z$ is a homomorphism with $K_\phi=\langle m\rangle$, where m is the characteristic of R. **6**

(b) Show that the set $U=\{a\in R/ab=ba$ for each $b\in R\}$ in a given ring R is a subring of R. Is U, a commutative subring of R ? Justify your answer. **6**

(c) Prove that an ideal $I=\langle K\rangle$ is a maximal ideal of a ring $(Z,+,\bullet)$ if and only if K is a prime number. **6**

OR

3 (a) Show that the subset $I=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d\in 2\mathbb{Z}\right\}$ is an ideal in a ring $(M_2(\mathbb{Z});+;\bullet)$. **6**

Moreover show that the quotient ring $\frac{M_2(\mathbb{Z})}{I}$ has exactly 16 elements.

(b) If E is a ring of even integers, then show that an ideal generated by 4 is a maximal ideal. **6**

(c) If $\phi:(R;+;\bullet)\rightarrow(R';\oplus;\odot)$ is a homomorphism then for an ideal I of R, prove that $\phi(I)$ is an ideal of $\phi(R)$. **6**

4 Attempt any two : 8

(a) If in a ring R, $x^2 = x$, $\forall x \in R$ then show that R is a commutative ring of characteristic 2.

(b) Show that an isomorphism between two rings is an equivalence relation.

(c) What is the intersection of a left ideal and right ideal in a ring R ? Justify your answer.

5 Attempt any two : 8

(a) Show that a division ring has no proper ideal.

(b) Is $M = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| b \in \mathbb{R} \right\}$ a maximal ideal in R ?

Justify your answer. Where $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$.

(c) Show that there can not be an integral domain with six elements.