HG-203 Seat No. ________

B. Sc. (Sem.-VI) Examination
March / April - 2015
Mathematics : CC-Math-601
(Abstract Algebra)

Time : 3 Hours] [Total Marks : 70

Instructions :(1) This question paper contains FIVE questions and all questions are compulsory.

(2) Figures to the right indicate the marks of each question.

1 (A) Define a ring. Show that the $\mathbb{Z}[i]=\{a+bi/a,b\in\mathbb{Z}\}$ is a commutative ring with unity under usual addition and multiplication. Moreover find unit elements in $\mathbb{Z}[i]$ **6**

(B) Give an example of a ring without unity, whose subring has a unity element. **6**

(C) Show that $(\mathbb{Z},+,\cdot)$ is a principal ideal ring. **6**

OR

1 (A) Prove that the characteristic of a ring R with unity is n if and only if n is the smallest positive integer with $n\cdot 1=0$. **6**

(B) If a commulative ring R with unity has no proper ideal then prove that R is a field. **6**

(C) If I_1 and I_2 are two ideals of a ring R then prove that I_1+I_2 is also an ideal of R. **6**

2 (A) Define a degree of a polynomial. For non-zero polynomials f and $g \in D[x]$, in usual notation, prove that : **6**

$[fg]=[f]+[g]$.

(B) State and prove Gauss's Lemma. **6**

(C) If the degree of a polynomial $f(x) \in F[x]$ is n, then prove that f(x) has at most n distinct roots in F. **6**

OR

2 (A) State and prove Eisenstein criteria. **6**

Deduce that $\sqrt[n]{P}$, $n \geq 2$ is always an irrational number for a prime P.

(B) Find the g.c.d. of $f(x) = 6x^3 + 5x^2 - 2x + 25$ and $g(x) = 2x^2 - 3x + 5 \in IR[x]$ and express it in the form $a(x)\, f(x) + b(x)\, g(x)$. **6**

(C) Suppose $f(x) \in F(x)$ and $a \in F$. Then prove that the remainder on dividing f(x) by x-a will be f(a). **6**

3 (A) Let R be a ring of rational numbers with odd denominators. then prove that **6**

(i) If I be a subset of R with even numerators then I is an ideal of R

(ii) R/I is a field.

(B) Prove that an ideal I in a commutative ring R with unity is a prime ideal if and only if R/I is an integral domain. **6**

(C) Define kernel of a homomorphism ϕ. If ϕ is a homomorphism of a ring R into a ring R' with kernel K_{ϕ} then show that K_{ϕ} is an ideal of R. **6**

OR

3 (A) Suppose R is a ring with unity. If $\phi : (R; +; .) \to (R'; \oplus; \odot)$ is a homomorphism, $\phi(1) \neq 0^1$. If R^1 is an integral domain then prove that $\phi(1)$ is the unity element of R^1. **6**

(B) If an ideal I in a commutative ring R with unity is a maximal ideal then prove that the quotient ring R/I is a field. **6**

(C) Show that the subset $I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big/ a, b, c, d \in 2\mathbb{Z} \right\}$ is an ideal in ring $(M_2(\mathbb{Z}); +; \cdot)$. **6**

Moreover show that the quotient ring $M_2(\mathbb{Z})/I$ has exact 16 elements.

4 Attempt any **TWO**. **8**

(1) If the characteristic of an integral domain D is P then prove that

$$(a+b)^P = a^P + b^P, \text{ for a, b } \in D$$

(2) Is division algorithm true in $\mathbb{Z}[x]$? Justify your answer.

(3) Prove that a homomorphism defined on a field is either a zero homomorphism or one-one.

5 Attempt any **TWO** : 8

(1) Define an integral domain.

Is $(M_2(\mathbb{Z});+;\cdot)$ an integral domain ?

Justify your answer.

(2) Obtain a polynomial equation in $\mathbb{Z}_5[x]$ whose solution set is $\{3,2\}$.

(3) For an ideal I of a ring R, prove that the quotient ring R/I is commulative if and only if ab–ba $\in I$, for a, b $\in R$.