PP-450 Seat No.________

B. Sc. (Sem. VI) Examination

April / May – 2016

CCMATH-602 : Mathematics Analysis-II

Time : 3 Hours] [Total Marks : 70

Instructions : (1) All questions are compulsory.
(2) Figures to the right side indicate marks of corresponding questions.

1 (a) Show that continuous image of connected set is connected. 6

(b) Let $f_1, f_2, f_3, \ldots\ldots, f_k$ be real functions on a metric space X. and $\bar{F}: X \to R^K$ defined by $\bar{F}(x) = (f_1(x), f_2(x), \ldots\ldots, f_k(x)),\quad x \in X$ then prove that $\bar{F}$ is continuous on X if and only if each function $f_1, f_2, \ldots, f_k$ is continuous on X. 6

(c) If f be a continuous real function on a metric space X and $Z(f) = \{P \in X \mid f(p) = 0\}$ prove that $Z(f)$ is closed set of X. 6

OR

1 (a) Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$. Prove that there exists a point $x \in [a,b]$ such that $f'(x) = \lambda$. 6

(b) Let f be monotonic on (a, b) then prove that the set of points of (a, b) at which f is discontinuous is at most contable. 6

(c) Using L' Hospital's rule evaluate following Limits : 6

(i) $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

(ii) $\lim_{x \to 0} \left(\frac{1}{x}\right)^{\tan x}$.

2 (a) Let f be a bounded real function on $[a,b]$ then prove that $f \in R(\alpha)$ on $[a,b]$ if and only if for every $\varepsilon > 0, \exists$ a partition P of $[a,b]$ such that $U(p, f, \alpha) - L(p, f, \alpha) < \varepsilon$. Where α is monotonically increasing function defined on $[\overline{a}, b]$. 6

(b) If $f \in R(\alpha)$ on $[a, b]$ and $a < c < b$ then prove : 6

(i) $f \in R(\alpha)$ on $[a, c]$ and $f \in R(\alpha)$ on $[c, b]$

(ii) $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$.

(c) Let $f : [1, 3] \to R$ be a function such that 6

$f(x) = 3$, $1 \le x < 2$

$= 5$, $2 \le x \le 3$ and

$\alpha : [1, 3] \to R$ be a function such that

$\alpha(x) = 0$, $1 \le x < 2$

$= 1$, $2 \le x \le 3$

then prove that $f \notin R(\alpha)$ on $[1, 3]$

OR

2 (a) If f is monotonic function on $[a, b]$ and α is continuous function on $[a, b]$ then prove that $f \in R(\alpha)$ on $[a, b]$. 6

(b) If $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$ on $[a, b]$ then prove that : 6

(i) $f \in R(\alpha_1 + \alpha_2)$ on $[a, b]$

(ii) $\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f d\alpha_2$.

(c) Let $a < x_0 < b$ and $f:[a,b] \to R$ be a function such that 6

$$f(x) = 1, \text{ if } x = x_0$$

$$= 0, \text{ if } x \neq x_0$$

Let α be an increasing function on $[a,b]$ which is continuous at x_0 then prove that

$$f \in R(\alpha) \text{ and } \int_a^b f\, d\alpha = 0.$$

3 (a) Let X be a metric space and $C(X)$ be the set of all complex valued continuous bounded function defined on X. Show that $C(X)$ is complete metric space. 6

(b) If K is compact and $f_n \in C(K), n = 1, 2, 3, ...$ If $\{f_n\}$ is pointwise bounded and equicontinuous on K then prove that $\{f_n\}$ is uniformly bounded on K. 6

(c) Show that $f(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n^3}$ is differentiable 6

for all x and $f'(x)=\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$.

OR

3 (a) Let $\{f_n\}$ be a sequence of function defined 6

on E. Prove that $\{f_n\}$ converges uniformly on E if and only if for every $\varepsilon > 0$, there exists a positive integer N such that

$m, n \geq N, x \in E \Rightarrow |f_n(x) - f_m(x)| < \varepsilon$

(b) Suppose $\{f_n\}$ is a sequence of function 6

defined on E and

$|f_n(x)| \leq M_n,\ \forall x \in E, n = 1, 2, 3, \ldots$ prove that

$\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

(c) Suppose $f_n(x)=\frac{1}{1+nx}, x\in(0,1), n=1,2,3,....$ **6**

then show that $f_n(x)\to 0$ pointwise on (0,1) but the convergence is not uniform.

4 Attempt any two : **8**

(i) If $f_n \in R(\alpha)$ on $[a,b]$ and if $f(x)=\sum_{n=1}^{\infty} f_n(x)$ $a\le x\le b$ and the series converges uniformly on $[a,b]$ then prove that

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} f_n d\alpha$$

(ii) Suppose X, Y, Z are metric space, $E\subset X$. Suppose $f: E\to Y$ is continuous at $P\in E$ and $g: f(E)\to Z$ defined by $h(x)=g(f(x)), x\in E$ then prove that h is continuous at p.

(iii) If $f\in R(\alpha)$ on $[a,b]$ and α is monotonically increasing function $[a,b]$ and $c>0$ then

$$f\in R(c\alpha) \text{ on } [a,b] \text{ and } \int_a^b f\,d(c\alpha)=c\int_a^b f d\alpha.$$

5 Attempt any two : **8**

(i) Show that $\sum \frac{\sin nx}{\sqrt{n}}$ converges uniformly on R.

(ii) Define $f(x) = x$ and $g(x) = x + x^2 \cdot e^{i/x^2}$ on (0, 1) show that L' Hospital rule fails in this case.

(iii) If $f(x) = 1$, if $x \in Q \cap [a, b]$

$=0$, if $x \notin Q \cap [a, b]$

and $\alpha(x) = x$, $x \in [a, b]$ then using definition of R-S integral prove that $f \notin R(\alpha)$ on $[a, b]$.