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GAF-453 Seat No._____

B. Sc. (Sem. V) Examination

November/December – 2015

Mathematics : Paper - CC - MATH - 501

(Group Theory)

Time : 3 Hours] [Total Marks : 70

1 (a) Let * be a binary operation on a finite set G. If * is associative and both cancellation laws holds in G then show that G is a group under *. (6)

(b) Prove that in a finite group G, O(a) / O(G) , for each a∈G. (6)

(c) Define a subgroup.For a commutative group G, show that H = {a∈G / O(a) is finite} is a subgroup of G. (6)

OR

1 (a) For a given n∈N, show that the set G = {[a]∈Z_n / (a,n) = 1} becomes a group under multiplication mod n with O(G) = $\phi(n)$, where ϕ is a Euler function. (6)

(b) In a group G, a∈G and O(a) = n. If for some positive integer q, (q, n) = 1 then prove that $O(a^q) = O(a)$. (6)

(c) State only Euler's and Fermat's theorem. Using this, show that if p is an odd prime then (1) $1^{p-1}+2^{p-1}+ \dots \dots +(p-1)^{p-1} \equiv (-1) \pmod p$

(2) $1^p+2^p+ \dots \dots +(p-1)^p \equiv 0 \pmod p$. (6)

2 (a) Prove that any two disjoint cycles in S_n are commutative. (6)

(b) Define normal subgroup. If K is a subgroup of G and H is a normal subgroup of G then show that K∩H is a normal subgroup of K. (6)

(c) Let G = C-{0} and G' = $\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} / a,b \in R, a^2+b^2 \neq 0\right\}$ be two groups under

multiplication. Define $f:G \to G'$ by $f(z) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$; for $z = a+ib \in C-\{0\}$

then show that f is an isomorphism. (6)

OR

2 (a) For $n \geq 2$, the set A_n of even permutations in S_n is a subgroup of order $n!/2$. (6)

(b) If a cyclic subgroup H of a group G is normal in G, then show that any subgroup of H is also normal in G. (6)

(c) Define: Transformation and Disjoint cycles. Express the permutation $f = (1\ 2\ 3\ 4)\ (1\ 2\ 4)\ (3\ 1\ 2)\ (5\ 6)$ in S_7 as a composition of disjoint cycles. Find O(f). (6)

3 (a) Suppose $G = \langle a \rangle$ is a finite cyclic group with $O(G) = n$. Show that for each $m \in N$, if m divides n then $\langle a^{\frac{n}{m}} \rangle$ is a unique subgroup of G of order m. (6)

(b) Prove that any two finite cyclic groups of the same order are isomorphic. (6)

(c) For a cyclic group $G = \langle a \rangle$ of order 16, obtain (1) order of subgroups generated by a^{12}, a^6, a^3. (2) all generators of group G. (6)

OR

3 (a) State and prove the Fundamental theorem of Homomorphism. (6)

(b) Show that an infinite cyclic group has exactly two generators. (6)

(c) Give an example of a finite abelian group of order 4 which is not cyclic. (6)

4 Attempt any Two : (8)

(a) In a commutative group G, $a,b \in G$ with $O(a) = m$ and $O(b) = n$. If $(m,n)=1$ then prove that $O(ab) = mn$.

(b) If H is a subgroup of group G with index 2 then prove that H is a normal subgroup of G.

(c) Show that every homomorphism image of an abelian group is abelian.

5 Attempt any Two : (8)

(a) In a group G, $a,b \in G$, if $a^5 = e$ with $a \neq e$ and $aba^{-1} = b^2$ then find O(b).

(b) Obtain the permutation fgf^{-1}, for $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$,

$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 2 & 3 & 1 & 6 \end{pmatrix} \in S_6$.

(c) If $\phi : G \to G'$ is onto isomorphism with $O(a) = n$, $a \in G$ then prove that $O(\phi(a)) = n$. Where G and G' are any two groups.
