# AAC-2112

Seat No.________

## M. Sc. (Sem. IV) Examination

**April - 2019**

## Mathematics : MTHP - 8

***(Algebra - II)***

Time : **3** Hours] [Total Marks : **90**

**1** Attempt any **two** : **18**

(a) If $L$ is a finite extension of $K$ and if $K$ is a finite extension of $F$, prove that $L$ is a finite extension of $F$.

(b) Let $F \subset K \subset E$ be three fields such that $K$ is an algebraic extension of $F$ and $\alpha \in E$ is algebraic over $K$. Show that $\alpha$ is algebraic over $F$.

(c) If $F$ is a field of characteristic 0 and if a, b, are algebraic over $F$, prove that there exists an element $c \in F(a,b)$ such that $F(a, b) = F(c)$.

(d) Let $Q$ be the field of rational numbers. Prove that $\sqrt{2}$ and $\sqrt{3}$ are algebraic over $Q$. Find the degree of :

(1) $Q(\sqrt{2})$ over $Q$.

(2) $Q(\sqrt{3})$ over $Q$.

(3) $Q(\sqrt{2}, \sqrt{3})$ over $Q$.

2 Attempt any **three** : 18

(a) State and prove remainder theorem.

(b) Prove that a polynomial of degree $n$ over a field can have at most $n$ roots in any extension field.

(c) Define the splitting field of a given polynomial $f(x) \in F[x]$. Construct the splitting field of $x^4 + 1 \in Q[x]$ and also find the degree of this splitting field over the field of rationals $Q$.

(d) Prove that if the polynomial $f(x) \in F[x]$ has a multiple root, then $f(x)$ and $f'(x)$ have a nontrivial (that is, of positive degree) common factor.

(e) Prove that it is impossible to trisect 60° using straightedge and compass alone.

(f) Prove that if $a$ and $b$ are constructible numbers, then $a \pm b$ are also constructible.

3 Attempt any **two** : 18

(a) Prove that if $K$ is a field an if $\sigma_1, \sigma_2, ...., \sigma_n$ are distinct automorphism of $K$, then it is impossible to find elements $a_1, a_2, ...., a_n$ not all 0, in $K$ such that

$a_1\sigma_1(u) + a_2\sigma_n(u) + ... + a_n\sigma_n(u) = 0$ for all $u \in K$.

(b) If $K$ is a finite extension of $F$, prove that $o(G(K, F)) \leq [K : F]$, where of $G(K, F)$ denotes the order of $G(K, F)$.

Compute $G(K, F)$, where $K$ is the field of complex numbers and $F$ is the field of real numbers.

(c) Prove that if $K$ is a normal extension of $F$, then K is the splitting field of some polynomial over $F$. Is $Q(\sqrt{2})$ normal over $Q$ ? Justify your answer.

(d) Express the following as polynomials in the elementary symmetric functions in $x_1, x_2, x_3$:

(1) $x_1^2 + x_2^2 + x_3^3$

(2) $x_1^3 + x_2^3 + x_3^3$

4 Attempt any **two** : **18**

(a) Prove that if $G$ is solvable group if and only if $G^{(k)} = (e)$ for some integer $k$. Is the symmetric group $S_4$ solvable ? Justify your answer.

(b) Prove that if $p(x) \in F[x]$ is solvable by radicals over F, then the Galois group over $F$ of $p(x)$ is a solvable group.

(c) Show that the polynomial $x^3 - 3x - 3$ over $Q$ is irreducible and have exactly two non real roots. Find its Galois group over $Q$.

(d) Show that for every prime number $p$ and for every positive integer $m$ there exists a field having $p^m$ elements.

5 Attempt any **six** : 18

(a) Find the degree of $\sqrt{2}+\sqrt{3}$ over the field of rationals $Q$.

(b) Is $\sqrt{2}+\sqrt[3]{5}$ algebraic over the field of rationals $Q$ ? Why.

(c) Is $Q\left(\sqrt[3]{2}\right)$ the splitting field of the polynomial $x^3-2\in Q[x]$ over the field of rationals $Q$ ? Why ?

(d) Is it possible to trisect 72° using straightedge and compass ? Why?

(e) Show that $Q\left(\sqrt{2}+\sqrt{3}\right)=Q\left(\sqrt{2}+\sqrt{3}\right)$.

(f) Show that any extension $E$ of a field $F$, such that $[E, F] = 2$, is a normal extension.

(g) Prove that a subgroup of a solvable group is solvable.

(h) State the fundamental theorem of Galois theory. (Do not prove)