ACA-3858 Seat No.________

M. Sc. (Sem. II) Examination

March/April – 2019

Mathematics : Paper - MTHP-2

(Algebra - I)

Time : **3** Hours] [Total Marks : **90**

Instructions :

(1) All questions are compulsory and carry equal marks.

(2) Follow the usual notations and conventions.

1 (a) If a group G is isomorphic to an external direct product of certain groups G_i, show that G is the internal direct product of groups $\overline{G}_i$ isomorphic to the G_i.

(b) If $O(G) = p^n$, where p is prime number then prove that $Z(G) \neq \{e\}$.

(c) Let p and q be primes with $p < q$. If p does not divide q-1, then show that any group of order pq is cyclic.

OR

1 (a) Show that any two isomorphic abelian groups of order p^n, where p is a prime, have the same invariants.

(b) If H and K are subgroups of a finite group G and $O(H) > \sqrt{O(G)},\ O(K) > \sqrt{O(G)}$ then prove that $H \cap K \neq \{e\}$.

(c) Show that any group of order $11^2 13^2$ must be an abelian group.

2 (a) State and prove the first part of Sylow's Theorem.

(b) Prove that S_{p^k} has a p-Sylow subgroup.

(c) Show that a group of order 56 is not simple.

OR

2 (a) If G is a finite group of order $O(G)$ and p is a prime number such that $O(G) = p^{\alpha} m$, where m is a positive integer not divisible by p, show that G has p-Sylow subgroups of G.

(b) Prove that the number of p-sylow subgroups of a finite group is of the form $1 + kp$, where p is a prime number.

(c) Prove that two abelian groups of order p^n are isomorphic iff they have the same invariants.

3 (a) An ideal $A = (a_0)$ is a maximal ideal of the Euclidean ring R if an only if a_0 is a prime element of R.

(b) Let R be a Euclidean ring and $a, b \in R$. Then prove that the greatest common divisor of elements a and b exists in R. Moreover $d = \lambda a + \mu b$, for some $\lambda, \mu \in R$.

(c) Prove that the ring of Gaussian integers is a Euclidean ring.

OR

3 (a) Show that every integral domain can be imbedded in a field.

(b) Let R be a Euclidean ring and let A be an ideal of R. Prove that there exists an element $a_0 \in A$ such that A consists of exactly all $a_0 x$, as x ranges over R.1.

(c) Prove that in a Euclidean ring R, $a \in R$ is unit of R if and only if $d(a) = d(1)$.

4 (a) If R is a unique factorization domain, then prove that $R[x]$ is also a UFD.

(b) If R is a unique factorization domain and $a, b \in R$, not both of them zero, show that a and b have a greatest common divisor $d(a, b)$ in R.

(c) Let R be a Euclidean ring. Then prove that every non-zero element is either a unit in R or can be written as a product of a finite number of prime elements of R.

OR

4 (a) State and prove the Eisenstein Criterion for irreducibility of polynomial.

(b) Define primitive polynomial. Prove that the product of two primitive polynomials over the field of rationals is a primitive polynimial.

(c) If R is an integral domain and F is its field of fractions, prove that every polynomial $f(x) \in F[x]$ can be written as

$$f(x) = \frac{g(x)}{a}, \text{ where } a \in R \text{ and } g(x) \in R[x].$$

5 Attempt any six :

(a) Prove that every abelian group is solvable.

(b) Prove that group of order 28 has a normal subgroup of order 7.

(c) Find the g.c.d. of $5+2i$ and $2-5i$ in the ring of Gaussion integers $Z[i]$.

(d) Show that group of order 36 is not simple.

(e) Is the polynomial $21-15x^2+111x^4+2x^5$ irreducible over the field of rational number ?

(f) Find all units of the ring $Z[i]$ and write all the associates of an element $a+ib \in Z[i]$.

(g) If R is a unique factorization domain and $f(x)$, $g(x)$ are in $R[x]$, them $c(f.g)=c(f).c(g)$. Where $c(f)$ is the content of $f(x)$.

(h) If R is a Unique factorization Domain, then prove that every non-zero member $f(x)$ of $R[x]$ can be written as $f(x)=d.f_1(x)$, where $d=c(f)$ and $f_1(x)$ is a primitive polynomial in $R[x]$.

(i) If p is a prime integer then prove that

$$(p-1)! \equiv (-1) \left(\text{mod } p\right).$$
