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AH-621

Seat No. ______

B. Sc. (Sem. VI) Examination

March - 2019

Mathematics : CC MATH-601

(Ad. Abstract Algebra)

Time : **3** Hours] [Total Marks : **70**

1 (a) Define principal ideal ring. Prove that the ring of integers $\langle Z, +, \times \rangle$ is a principal ideal ring. **6**

(b) Let R bearing, the set $U = \{a \in R / ab = ba, \forall b \in R\}$ is a commutative subring of R. Is U ideal of R ? Verify. **6**

(c) Prove that the characteristic of a ring R with unity is n if and only if n is the smallest positive integer with $n.1 = 0$. **6**

OR

1 (a) Define characteristic of ring R. Prove that the characteristic of an integral domain is either zero or prime number. **6**

(b) If in a ring $R, x^2 = x, \forall x \in R$, then show that R is a commutative ring of characteristic 2. **6**

(c) For a given element a, of a ring R, the set $A = \{ba + na / b \in R, n \in Z\}$. **6**

(1) Show that A is left ideal of R.

(2) If I is any left ideal of R with $a \in I$, then $A \subset I$.

(3) If ring R is commutative, then $A = \langle a \rangle$.

2 (a) State and prove the 'Division Algorithm' for polynomials. **6**

(b) If $f(x), g(x) \in F[x]$ are not both zero polynomials, then prove that their g.c.d. $d(x)$ exists and $d(x) = a(x) \cdot f(x) + b(x) \cdot g(x)$, for some polynomials $a(x), b(x) \in F[x]$. **6**

(c) Prove that the polynomial $1 + x + x^2 + x^3 + ... + x^{p-1}$ where p being prime, is irreducible over the field of rational numbers. **6**

OR

2 (a) Prove that the polynomial ring $F[x]$ is a principal ideal domain. **6**

(b) State and prove Eisenstein Criterion.

(c) Find g.c.d of $f(x) = x^3 + 3x^2 + 3x + 3$ and $g(x) = 4x^3 + 2x^2 + 2x + 2 \in Z_5[x]$ and express it in the form $a(x) \cdot f(x) + b(x) \cdot g(x)$. **6**

3 (a) Prove that a homomorphism defined on the ring $(Z, +, \times)$ is either zero homomorphism or identity mapping. **6**

(b) Prove that an ideal I in a commutative ring R with unity is a maximal ideal if and only if the quotient ring $\frac{R}{I}$ is a field. **6**

(c) In ring $(Z_{12}; +_{12}; \times_{12})$, find its all prime and maximal ideals. **6**

OR

3 (a) Prove that an ideal $I = \langle p \rangle$ is a maximal ideal of ring $\langle Z, +, . \rangle$ if and only if p is prime. **6**

(b) Let I be an ideal of a ring R. Prove that in quotient ring $\frac{R}{I}$, the product $(I + a) \cdot (I + b) = \mathrm{I} + ab$ is well defined product, for every $a, b \in R$. **6**

(c) If $\phi : (R; +, *) \to (R'; \oplus, \otimes)$ is a homomorphism then, prove that for an ideal U of $\phi(R')$, $\phi^{-1}(U)$ is an ideal of R. **6**

4 Attempt any **two** : **8**

(a) Prove that every finite integral domain is a field.

(b) Show that the subset

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / a, b, c, d \in 2Z \right\}$$ is an ideal of ring

$(M_2(Z); +, .)$. Also write the elements of quatient ring $M_2(z)/_I$.

(c) Show that $I = \{ f(x) \in Z[x] /$ coefficients of $f(x)$ are even integers$\}$ is a prime ideal in $Z[x]$. Will I be a maximal ideal ?

5 Attempt any **two** : 8

(a) Give an axample of the following :

(1) Non-commutative ring but its subring is commutative.

(2) Division ring which is not a field.

(b) Examine the following polynomials for irreducibility over Q.

(1) $f(x) = x^3 + 9x + 3$

(2) $g(x) = x^4 + x^3 + x^2 + x + 1$

(c) Show that ideal $I = \langle x^3 - x - 1 \rangle$ is a maximal ideal in $Z_3[x]$.
