BB-210 Seat No.________

B. Sc. (Sem. VI) Examination

March / April – 2014

Mathematics : CC Math : 602

(Analysis-II)

Time : 3 Hours] [Total Marks : 70

Instructions : (1) All questions are compulsory.

(2) Figure to the right side indicate marks of corresponding questions.

1 (a) Prove that mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}v$ is open set in X, for every open set v in X. **6**

(b) Suppose f is a continuous mapping of a compact metric space X into a metric space Y then prove that $f(X)$ is compact. **6**

(c) If $X=[0,1]$ and $f(x)=\frac{1}{x}$ then show that $f:X\to R$ is continuous on X but not uniformly continuous on X. **6**

OR

1 (a) Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$, g is defined on an interval I which contains the range of f and g is differentiable at point $f(x)$. If $h(t) = g(f(t))$, $a \le t \le b$ then prove that h is differentiable at x and $h'(x) = g'(f(x)) \cdot f'(x)$. 6

(b) Let f be monotonic on (a, b) then prove that the set of points (a, b) at which f is discontinuous is at most countable. 6

(c) If $f(x) = |x|^3, x \in R$ then compute $f'(x), f''(x), \ \forall x \in R$ and show that $f^{(3)}(0)$ does not exist. 6

2 (a) State and prove the fundamental theorem of calculus. 6

(b) If $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$ on $[a,b]$, then prove that $f_1 + f_2 \in R(\alpha)$ on $[a,b]$ and

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 d\alpha$$

(c) Prove that $\int_{0}^{3} x \, d(x-[x]) = \frac{-3}{2}$. **6**

OR

2 (a) If f is a monotonic function on $[a,b]$ and α is continuous on $[a,b]$ then prove that $f \in R(\alpha)$ on $[a, b]$. **6**

(b) Suppose : **6**

(i) $C_n \geq 0$ for n = 1,2,3,.....

(ii) $\sum C_n$ converges

(iii) $\{S_n\}$ is a sequence of distinct points in (a, b)

(iv) $\alpha(x) = \sum_{n=1}^{\infty} C_n \, I(x - S_n)$

(v) f is continuous on $[a, b]$

then prove that $\int_{a}^{b} f d\alpha = \sum_{n=1}^{\infty} C_n \, f(S_n)$.

(c) Define f on $[a, b]$ as 6

$$f(x) = \frac{1}{2^n}, \quad \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, n = 1, 2, 3 ....$$

$$f(0) = 0$$

and $\alpha(x) = x, \ \forall\, x \in [0, 1]$ then prove that $f \in R(\alpha)$ on $[0, 1]$ and $\int_0^1 f(x)\,dx = 2/3$.

3 (a) Suppose K is compact suppose a sequence $\{f_n\}$ of continuous functions on k converges pointwise to a continuous functions f on k and $f_n(x) \ge f_{n+1}(x), \ \forall x \in k$, n = 1, 2, 3,... then prove that $f_n \to f$ uniformly on k. 6

(b) Let X be a metric space and $\mathcal{C}(X)$ be the set of all complex valued continuous bounded functions defined on X, then show that $\mathcal{C}(X)$ is complete metric space. 6

(c) Prove that sequence $\{f_n\}$ of functions defined on [0, 1] is uniformly bounded on [0, 1] but not uniformly converges on [0, 1]. **6**

Where $f_n(x) = \frac{x^2}{x^2 + (1-nx)^2}, 0 \le x \le 1, n = 1, 2, 3, ...$

OR

3 (a) If K is a compact metric space. **6**

$f_n \in \mathcal{C}(K), \quad n = 1, 2, 3, ...$ and $\{f_n\}$ converges uniformly on K then prove that $\{f_n\}$ is an equicontinuous on K.

(b) Let α be monotonically increasing on $[a, b]$ **6**

suppose $f_n \in R(\alpha)$ on $[a, b]$, $n = 1, 2, 3, ...$

and suppose $f_n \to f$ uniformly on [a, b] then

$f \in R(\alpha)$ on $[a, b]$ and

$$\int_a^b f \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha$$

(c) If $f_n(x) = nx(1-x)^n$, $x \in [0,1], n = 1, 2, 3, \ldots$ **6**

then prove that $\{f_n\}$ is not uniformly converges on [0, 1]

4 Attempt any **two** : **8**

(a) Suppose X, Y, Z are metric space, $E \subset X$. Suppose $f : E \to Y$, $g : f(E) \to Z$ and $h : E \to Z$ defined by $h(x,) = g(f(x)), x \in E$. If f is continuous at $P \in E$ and g is continuous at $f(P)$ then prove that h is continuous at P.

(b) If $f \in R(\alpha)$ on $[a,b]$ and C is positive constant then prove that $f \in R(C\alpha)$ and

$$\int_a^b f \, d(C\alpha) = C \int_a^b f \, d\alpha.$$

(c) Suppose $\{f_n\}$ is a sequence of functions defined on E and suppose $|f_n(x)| \le M_n$, $x \in E, n = 1, 2, 3 \ldots$.

then prove that $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

5 Attempt any two : 8

(a) State the types of discontinuity for the following functions :

(i) $f(x) = x+2 \quad , 0 \le x < 1$

$= 3x+1 \; , 1 \le x \le 2$ at $x = 1$

(ii) $f(x) = x - [x]$ at $x = 3$.

(b) If $f(x) = [x]$ and $\alpha(x) = x$, then prove that $f \in R(\alpha)$ on $[0, 2]$ and also find $\int_0^2 f d\alpha$.

(c) Show that $\sum \frac{\sin n x}{\sqrt{n}}$ converges uniformly on R.