



MAV-3468

Seat No. _____

M. Sc. (Sem. III) Examination

October / November - 2018

Mathematics : Paper -MTHP-VI

(Functional Analysis-I)

Time : 3 Hours] [Total Marks : 90

- Instructions :**
- (1) All questions are compulsory and carry equal marks.
 - (2) Follow standard notations and conventions.

1 Attempt any **three** : 18

- (a) Let a linear space L be the sum of two subspaces M and N . Prove that $L = M \oplus N$ if and only if $M \cap N = \{0\}$.
- (b) Let M be a closed linear subspace of a normed linear space N . Show that N/M is a Banach space if N is a Banach space.
- (c) Let X be a normed space, Y be a closed subspace of X and $Y \neq X$. Let r be a real number such that $0 < r < 1$. Then show that there exist some $x_r \in X$ such that $\|x_r\| = 1$ and $r \leq \text{dist}(x_r, Y) < 1$.

(d) If S is a linearly independent set of vectors in a linear space L then show that there exists a basis B for L such that $S \subseteq B$. Hence deduce that every non-zero linear spaces has a basis.

(e) Show that any non-zero infinite dimensional linear space of dimension n is isomorphic to R^n or C^n .

2 Attempt any **three** :

18

(a) Show that there exist a linear functional f on ℓ^∞ such that $\|f\|=1=f(a)$ and $f(\tau(x))=f(x)$

for all $x \in \ell^\infty$ where $a=(1,1,\dots)$ and $\tau(x)(j)=x(j+1)$ for $j=1,2,\dots$. Also deduce

that such functional f satisfies $f(x)=\lim_{j \rightarrow \infty} x(j)$

so that f is a Banach limit.

(b) Let M be a closed linear subspace of a normed linear space N and $x_0 \in N-M$. If d is a distance from x_0 to M then show that there exists a functional f^* such that $f(M)=0$,

$f(x_0)=1$ and $\|f^*\|=1/d$.

- (c) State and prove Hahn-Banach theorem.
- (d) Let X and Y are Banach space and $F: X \rightarrow Y$ is linear map which is closed and surjective. Then show that F is continuous and open.
- (e) Prove that a normed linear space N is separable if its conjugate space N^* is.

3 Attempt any **three** :

18

- (a) State and prove the closed graph theorem. Give an example to show that this theorem may not hold if the normed linear spaces X and Y are not Banach spaces.
- (b) Illustrate projection on a Banach space with explicit definition. Is it always possible that projection induces the decomposition of a Banach space ? Justify your answer in detail.
- (c) Show that closed graph theorem can be derives as a consequences of open mapping theorem whenever the open map is one-one.
- (d) Define the conjugate of an operator of norm linear space and show that it is only isometry.

- (e) Show that closed graph theorem can be derives as a consequence of open mapping theorem whenever the open map is one one.

4 Attempt any **three** :

18

- (a) Let E be a non-empty closed convex subject of a Hilbert space H , then prove that for each $x \in H$, there exists a unique best approximation from E to x . In particular there is a unique element in E of minimal norm.

- (b) State and prove projection theorem.

- (c) Prove Holder's and Minkoski's inequality for $l_p (1 \leq p \leq \infty)$.

- (d) For a Hilbert space H show that the equivalence of the following;

for a orthonormal set $\{e_i\}$

- (i) $\{e_i\}$ is complete.

- (ii) $x \perp \{e_i\} \Rightarrow x = 0$

- (iii) If x is an arbitrary vector in H then

$$x = \sum \langle x, e_i \rangle e_i.$$

- (iv) If x is an arbitrary vector in H then

$$\|x\|^2 = \sum |\langle x, e_i \rangle|^2.$$

- (e) If B is complete Banach space whose norm obeys (i) parallelogram law (ii) polarization identity, then show that B is a Hilbert space.

5 Attempt any six :

18

(a) Establish :

- (i) Parallelogram law
(ii) Pythagorean principle

(b) Let X be a norm space and Y be a subspace of X then (a) for $x \in X, y \in Y$ and $k \in K$, then prove that $\|kx + y\| > |k| \text{ dist. } (x, Y)$.

(c) Let X be a Banach space, Y be a norm space and (F_n) be a sequence in $BL(X, Y)$ such that the sequence $(F_n(x))$ converges in Y for every $x \in X$. For $x \in X$ defined $F(x) = \lim_{n \rightarrow \infty} F_n(x)$. And if E be a totally bounded subset of X , then prove that $(F_n(x))$ converges to $F(x)$ uniformly for $x \in E$.

- (d) Let Y be a closed subspace of a normed space X for $x+Y$ in the quotient space X/Y , Let $|x+Y| = \inf \{|x+y| : y \in Y\}$, then prove that $\|\cdot\|$ is a norm on X/Y is a quotient norm.
- (e) Prove that every finite dimensional subspace Y of a norm space X is closed in X . If M is a linear subspace of Hilbert space H , then show that M^\perp is closed linear subspace of H .
- (f) Show that M is closed in H if and only if $M = M^{\perp\perp}$.
- (g) Let P be a projection on a Banach space X . Show that $Y = R(P)$ and $Z = N(P)$ are closed subspace of X and $X = Y \oplus Z$.
- (h) By mean of natural imbedding, show that T^{**} is an extension of T , where T is an operator on a nls N . when does $T^{**} = T$?

(i) Prove or disprove :

$$(i) \quad x_n \rightarrow x_0 \Rightarrow \|x_n\| \rightarrow \|x_0\|$$

$$(ii) \quad x_n \rightarrow x \text{ and } y_n \rightarrow y \Rightarrow x_n + y_n \rightarrow x + y.$$
