MEB-4695 Seat No.________

M. Sc. (Sem. ~~III~~ I) Examination

November / December - 2018

MTHP - I : Mathematics

(Measure Theory)

Time : **3** Hours] [Total Marks : **90**

Instructions : (1) All questions are compulsory and carries equal marks

(2) Follow standard notations and conventions.

1 Attempt any **three** : **18**

(a) Show that the Cantor set and all its subsets are measurable and each of them has a measure zero.

(b) *Let* $\{An\}$ be a countable collection of sets of real numbers. Prove that $m^*(\cup A_n) \le \sum m^*(A_n)$.

(c) Prove that the family m of measurable sets is an algebra of sets.

(d) Prove that for any real a, the interval (a, ∞) is measurable.

(e) For given set $A \,\&\, \in > 0$ show that there exists an open set O containing A and G_δ set G containing A such that

$$m^* O \le m^* A + \in;. \; and \; m^* A = m^* G.$$

2 Attempt any **three** : **18**

(a) $Let\{f_n\}$ be a sequence of measurable functions defined on a finite measurable set $E. Let\ f$ be–valued function such that for each x in E, $f_n(x) \to f(x)$. Then prove that for given $\varepsilon > 0$ and $\delta > 0$, there is a measurable subset A of E with $m^*(A) \le \delta$ and an integer N such that for all x not belonging in A and all $n \ge N, |f_n(x) - f(x)| < \varepsilon$.

(b) *Let f* be a nonnegative integrable function. Show that the function F defined by $F(x) = \int_{-\infty}^{x} f(x)\,dx$ is continuous.

(c) State Littele Wood's three principles. Prove any one of them.

(d) f is measurable then show that $|f|$ is. Is the converse true ? Justify your Answer.

(e) Show that the product of two absolutely continuous functions is absolutely continuous.

3 (a) State and prove Fatou's lemma. Also give an example to have strict inequality in Fatou's Lemma and another example to have equality in Fatou's Lemma.

(b) *Let f be* a nonnegative measurable function. Prove that $\int f(x)\,dx = 0$ implies $f = 0$ a.e.

(c) Show that the Monotone Convergence Theorem need not hold for a decreasing sequence of functions.

OR

3 (a) State and prove Bounded Convergence theorem.

(b) Give an example of a Lebesgue integrable function that is not Riemann integrable.

(c) *Let* $\{f_n\}$ be a sequence of measurable function that converges in measure to a function *f*. Show that there exists a sub-sequence $\{f_{n_k}\}$ that converges to *f* a.e.

4 (a) If *f* is bounded and measurable on $[a,b]$ *and* $F(x)-F(a)=\int_a^x f(t)\,dt$ then show that $F'(x)=f(x)$ for almost all *x* in $[a,b]$.

(b) If *f* is of bounded Variation on $[a,b]$ then show that $\int_a^b |f'| = T_a^b(f)$.

(c) If *f* is absolutely continues on $[a,b]$ and $f'(x)=0$ almost everywhere then prove that *f* is Constant.

OR

4 (a) If $a \le c \le b$ and *f* is bounded variation on $[a,b]$ then show that $T_a^b(f)=T_a^c(f)+T_c^b(f)$.

(b) If *f* is absolutely continues on $[a,b]$ and $f'(x)=0$ almost everywhere then prove that *f* is constant.

(c) If *f* is integrable on $[a,b]$ and $\int_a^x f(t)\,dt=0$ for all $x\in[a,b]$ then show that $f(t)=0$ a.e. in $[a,b]$

5 Do any **six** in briefly :

(a) Define $P_a^b(f), N_a^b(f)$ *and* $T_a^b(f)$. Show that $f(b)-f(a)=p-n$.

(b) Show that uniform convergence $\Rightarrow$ convergence in measure. Is the converse true ? Justify.

(c) Give an example of a function which lebesgue integrable but not Riemann integrable.

(d) If $f=0$ a.e. then $\int f=0$. Is the converse true ? Justify.

(e) Let A be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a finite collection of open intervals covering A. Show that $m^*(A)>=1$.

(f) Show that the set of rational is F_6-set. Is the set a G_8-set? Explain.

(g) Let A be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a finite collection of open intervals covering A. Find the range of $\Sigma 1(I_n)$.

(h) If $f \in B_v[a,b]$ then show that f is a bounded function.

(i) Give an example of an uncountable set; which is not an interval of measure 2018.

(f) For any two disjoint subsets A and B of R, show that $\chi_{A\cup B}=\chi_A+\chi_B$.