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AAL-7331

Seat No._________

M. Sc. (Sem. II) Examination

April / May - 2018

Mathematics : MCB - II

(Advanced Linear Algebra)

Time : 2 Hours] [Total Marks : 35

Instructions : (1) Each the questions is compulsory and carry equal marks.

(2) Follow the standard notations and conventions.

Note : Throughout the question paper assume that V is an n-dimensional vector space over a field F, $B=\{v_1, v_2, \ldots\ldots, v_n\}$ is an ordered basis of V over F and $T \in A_F(V)$.

1 (a) Define : Idempotent and Nilpotent linear transformations on an n-dimensional vector space V over a field F. Show that :

(1) If $T \in A_F(V)$ is idempotent, then $V = V_0 \oplus V_1$, where $V_0 = ker(T)$ and $V_1 = ker(T - I)$.

(2) If $S \in A_F(V)$ is nilpotent, then a characteristic roots of T is 0.

(b) Show that an algebra A, with unit element, over a field F is isomorphic to a subalgebra of the algebra A_F (V), for some vector space V over F.

OR

(b) If $\lambda_1, \lambda_2, \lambda_3, ... \lambda_k$ in F are the distinct characteristic roots of T and $v_1, v_2, v_3, ... v_n$ are characteristic vectors of T belonging to $\lambda_1, \lambda_2, \lambda_3, ... \lambda_k$ respectively, for some integer $k \leq n$, show that $v_1, v_2, v_3, ... v_n$ are linearly independent over F.

2 (a) If T has all its characteristic roots in F, show that there is a basis of V over F in which the matrix of T in the basis B, $m(T) = [T]_B$, is triangular matrix, and T satisfies a polynomial of degree n over F.

(b) Show that the matrices of T in two bases of V over F are similar in F_n.

OR

(b) Let F be a field with char $F \neq 2$, and $V = F^{(3)}$ be the vector space over F. If

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 0 & -11 \\ 0 & 1 & 6 \end{bmatrix} \in F_3,$$

then determine its minimal polynomial over F and its similar matrix in F_3 which is a diagonal matrix.

3 (a) Define $tr(T)$ and establish that it is independent from choice of basis. Show that $tr(T)$ is the sum of the characteristic roots of T.

OR

(a) If T is i nilpotent of index n_1, prove that there exists a basis B of V such that the matrix of T in this basis has the form diagonal matrix $diag\left[M_{n_1} \; M_{n_2} \; .... \; M_{n_r}\right]$ in Fn, Where $n_1 \geq n_1 \geq ...... \geq n_1$ with $n_1 + n_1 + ... + n_1 = n$.

(b) For a matrix $A \neq 0$ in F_n, where $F = \square$, prove that $tr\ (A^*A) > 0$ and characteristic roots of a Hermitian matrix are all real.

3 (a) If all the distinct characteristic roots $\lambda_1, \lambda_2, \lambda_3, \ldots. \lambda_k$ of T lie in F, show that V can be written as $V = V_1 \oplus V_2 \oplus \ldots.. \oplus V_k$, where each subspace $V_j = \ker\left(\left(T - \lambda_j I\right)^{l_j}\right)$ and the linear transformation T_j, induced by T on V_j, has only one characteristic root λ_j, on V_j.

OR

(a) If $char(F) = 0$, and if $tr\left(T^j\right) = 0$ for all $j \geq 1$, prove that T is nilpotent.

(b) If $A \in F_n$, prove that $A = B + C$ where $B \in F_n$ is symmetric matrix and $C \in F_n$ is skew-symmetric matrix are uniquely determined.

4 (a) For $A \in F_n$ prove that $det(A) = det(A')$.

(b) Find the Jordan canonical form of any one of the matrix $A = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix} \in F_3$.

OR

(b) Show that the characteristic roots of $A \in F_n$ are the roots with correct multiplicity of the equation $det(A - xI) = 0$.

5 Answer Briefly any **three** :

(a) Prove that T is invertible if and only if it maps V onto V.

(b) Let $F = \square$ and $V = \square^2$ and $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \in F_2$ be the matrix of T relative to the standard ordered basis of V over F. Prove that the V and zero subspace are only the subspaces of V which are invariant under T.

(c) For $T, S \in A_F(V)$, show that $r(TS) \le r(T)$ and $r(TS) = r(ST)$ whenever S is regular.

(d) If $A \in F_n$ is invertible, prove that $\det(A) \ne 0$.

(e) Let $F = \mathbb{R}$ and $V = \mathbb{R}^2$, find the trace and determinant of T defined as $T(a, b) = (a, a+b)$.
