AS-1910 Seat No. ________

B. Sc. (Sem. VI) Examination

March / April – 2018

Mathematics : CCMAT-602

(Analysis - II)

Time : **3** Hours] [Total Marks : **70**

Instructions : (1) All questions are compulsory.

(2) The figure to right indicate the marks of the corresponding question.

1 (a) A mapping f of a metric space X into metric Y is continuous on X if and only if $f^{-1}(v)$ is open in X, for every open set v in Y. **6**

(b) Let f be monotonically increasing on (a, b) then prove that $f(x+)$ and $f(x-)$ exists at every point of x of (a, b). **6**

(c) Using L'Hospital Rule evaluate following limit : 6

(i) $\lim_{x\to 0} \dfrac{x-\tan x}{x^3}$

(ii) $\lim_{x\to 0} \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$

OR

1 (a) If f is a real continuous function on $[a,b]$ which is differentiable in (a,b), then there is a point $x\in(a,b)$ at which 6

$f(b)-f(a)=(b-a)f'(x).$

(b) Show that continuous image of compact is compact. 6

(c) If $f(x)=x^2$ then prove that f is continuous on R but not uniformly continuous. 6

2 (a) If $f\in R(\alpha_1)$ and $f\in R(\alpha_2)$ then prove : 6

(i) $f\in R(\alpha_1+\alpha_2)$

(ii) $\int_a^b f\ d(\alpha_1+\alpha_2)=\int_a^b f\ d\alpha_1+\int_a^b f\ d\alpha_2$

(b) Define Upper Integral and Lower integral. **6**

Prove that :

$$\underline{\int_a^b} f d\alpha \le \overline{\int_a^b} f d\alpha$$

(c) Suppose α increases on $[a,b]$, $a \le x_0 \le b$, α is continuous at x_0, $f(x_0) = 1$ and $f(x) = 0$ if $x \ne x_0$ then prove that $f \in R(\alpha)$ and $\int_a^b f d\alpha = 0$. **6**

OR

2 (a) Suppose : **6**

(1) $C_n \ge 0$ for 1,2,3,..........

(2) $\sum C_n$ converges

(3) $\{S_n\}$ is a sequence of distinct Points in (a,b) and $\alpha(x) = \sum C_n \cdot I(x - S_n)$.

If f be continuous on $[a,b]$ then prove that

$$\int_a^b f \, d\alpha = \sum C_n f(S_n).$$

(b) If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$ then prove that : **6**

(i) $fg \in R(\alpha)$ on $[a, b]$

(ii) $|f| \in R(\alpha)$ and $\left|\int_a^b f \, d\alpha\right| \leq \int_a^b |f| \, d\alpha$

(c) Let f be a continuous function on $[a, b]$ and **6**

$$\alpha : [a, b] \to R, \ \alpha(x) = 0, \ 0 \leq x \leq 1$$

$$= c, \ x = 1$$

$$= 1, \ 1 \leq x \leq 2 \text{ where } 0 \leq c \leq 2$$

Then prove that :

$$f \in R(\alpha) \text{ on } [0, 2] \text{ and } \int_0^2 f d\alpha = f(1).$$

3 (a) Suppose $f_n \to f$ uniformly on E in a metric space X. Let x be a limit point of E and suppose that $\lim_{n \to \infty} f_n(t) = A_n$ then prove that $\{A_n\}$ convergence and $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$. In other words prove that **6**

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{t \to x} \lim_{n \to \infty} f_n(t).$$

(b) If K is a compact metric space if $f_n \in C(K)$, $n = 1, 2, 3, \ldots\ldots$ and if $\{f_n\}$ convergence uniformly on K then $\{f_n\}$ is equi-continuous on K. **6**

(c) Prove that sequence does not convergent uniformly on R, where $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $x \in R$ and $n = 1, 2, 3, \ldots\ldots$ **6**

OR

3 (a) Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in R(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \ldots\ldots$ And suppose $f_n \to f$ uniformly on $[a. b]$ then prove that $f \in R(\alpha)$ on $[a, b]$ and **6**

$$\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha.$$

(b) Prove that the series $\sum f_n(x)$ of functions defined on E convergence uniformly on E if and only if for every $\varepsilon > 0$, $\exists$ a positive integer N such that $m \geq n \geq N \Rightarrow \left| \sum_{k=n}^{m} f_k(x) \right| < \varepsilon$, $x \in E$. **6**

(c) If $f_n(x) = \frac{1}{1+nx}$, $x \in (0,1)$ then show that $f_n(x) \to 0$ point wise on $(0,1)$ but the convergence is not uniform. 6

4 Attempt any two : 8

(i) Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \alpha < f'(b)$. Then there is a point $x \in (a,b)$ such that $f'(x) = \alpha$.

(ii) If $P*$ is a refinement of P then prove $L(P, f, \alpha) \le (P*, f, \alpha)$.

(iii) Suppose $\lim_{n \to \infty} f_n(x) = f(x)$, $\forall x \in E$ and $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ then prove that $f_n \to f$ uniformly on if and only if $M_n \to 0$ as $n \to \infty$.

5 Attempt any two : 8

(i) Define $f(x) = x + 2 \quad -3 < x < -2$

$$= x + 2 \quad 0 \le x < 1 = \quad -x - 2 - 2 \le x \le 0$$

Discuss the continuity at $x = 0$.

(ii) Give an example of a sequence of functions for which $\lim_{n\to\infty}\left[\int_0^1 f_n(x)\,dx\right] \neq \int_0^1\left[\lim_{n\to\infty} f_n(x)\right]dx$.

(iii) If $f(x)=x^3$, $\alpha(x)=x$, $\forall x \in [0, a]$ then prove that $\int_0^a f\,d\alpha = \frac{a^4}{4}$.
