AR-1903 Seat No. ______

B. Sc. (Sem. VI) Examination

March / April - 2018

Mathematics : Paper - CC-MATH-601

(Abstract Algebra)

Time : 3 Hours] [Marks : 70

1 (a) Prove that a non-zero element m of a ring $(Z_n, +_n, \times_n)$ is a zero divisor if and only if m and n are not relatively prime. Obtain all the zero divisors of ring $(Z_{12}, +_{12}, \times_{12})$ 6

(b) If I_1 and I_2 are two ideals of a ring R, then show that $I_1 \cup I_2$ is an ideal of R if and only if either $I_1 \subset I_2$ or $I_2 \subset I_1$. 6

(c) Let R be a ring, prove that set $U = \{a \in R / ab = ba, \forall b \in R\}$ is a commutative subring of R. 6

OR

1 (a) Prove that the ring of integers $(Z, +, \bullet)$ is a principal ideal ring. 6

(b) The product $I_1 \cdot I_2$ of two ideals I_1 and I_2 in a ring R is defined as, 6

$$I_1 \cdot I_2 = \left\{ \sum_{i=1}^{n} a_i b_i / a_i \in I_1, b_i \in I_2; n \in N \right\}$$

then, show that $I_1 \cdot I_2$ is an ideal of ring R and $I_1 \cdot I_2 \subseteq I_1 \cap I_2$.

(c) Prove that if a commutative ring R with unity has no proper ideal then R is a field. 6

2 (a) State and prove the 'Division Algorithm' for polynomials. 6

(b) Define the degree of a non-zero polynomial f defined on an integral domain D. Show that $0 \le \deg(f) \le \deg(f.g)$, for non-zero polynomials f and g defined on an integral domain D. 6

(c) Find g.c.d. of $f(x) = x^3 - 4x^2 + 4x - 3$ and $g(x) = x^3 - 3x^2 + 2x - 6$. Find $m(x)$ and $n(x)$ such that g.c.d. $= m(x) \cdot f(x) + n(x) . g(x)$. 6

OR

2 (a) Prove that the polynomial ring $F[x]$ is a principal ideal domain. 6

(b) State and prove Eisenstein Criterion. 6

(c) Find g.c.d. of $f(x) = x^4 + x^3 - 3x^2 - x + 2$ and $g(x) = x^4 + x^3 - x^2 + x - 2$ over the field Q of rational numbers. Find $m(x)$ and $n(x)$ such that g.c.d. $= m(x) \cdot f(x) + n(x) \cdot g(x)$. 6

3 (a) Prove that an ideal $I = \langle p \rangle$ is a maximal ideal of ring $\langle Z, +, \bullet \rangle$ if and only if p is prime. 6

(b) Let $\langle R ; + ; \bullet \rangle$ be a ring with unity. Prove that the mapping $\phi : (Z ; + ; \bullet) \to (R ; + ; \bullet)$ define by $\phi(n) = n.1, n \in Z$ is a homomorphism with $K_{\phi} = \langle m \rangle$, where m is the characteristic of R. 6

(c) If E is a ring of even integers, then show that an ideal generated by 4 is a maximal ideal. **6**

OR

3 (a) Prove that an ideal I in a commutative ring R with unity is a maximal ideal if and only if the quotient ring R/I, is a field. **6**

(b) If $\phi : (R; +, *) \rightarrow (R'; \oplus, \otimes)$ is a homomorphism and I is an ideal of ring R then prove that $\phi(I)$ is an ideal of $\phi(R')$. **6**

(c) In ring $(Z_{12}; +_{12}; \times_{12})$, find its all prime and maximal ideals. **6**

4 Attempt any **two** : **8**

(a) If for given elements a and b of a ring R with unity, $1-ab$ is a unit element then so is $1-ba$.

(b) State and prove the Remainder theorem.

(c) Let I be an ideal of a ring R. Prove that in R/I, the product $(I+a)\cdot(I+b) = I+ab$ is well defined product, for every $a, b \in R$.

5 Attempt any **two** : **8**

(a) Give an example of the following :

(1) Neither a commutative ring R nor has a unit element.

(2) Non-commutative ring with a finite number of elements.

(3) Commutative ring which is not an integral domain.

(b) Find whether the polynomial $x^2 + x + 4$ is irreducible over the field $Z_{11}[x]$ or not.

(c) Show that ideal $I = \left\langle x^3 - x - 1 \right\rangle$ is a maximal ideal in $Z_3[x]$.
