KQ-1625 Seat No. ______

M. Sc. (Sem. I) Examination
November / December - 2017
Mathematics : MTHP-1
(Measure Theory)

Time : **3** Hours] [Marks : **90**

Instructions : (1) There are five questions; all questions are compulsory.
(2) Each question carries 18 marks.
(3) Follow standard notations and conventions.

1 Attempt any **three** : **18**

(a) Define algebra of sets.

Prove : If $\mathcal{C}$ is a collection of subset of X; then there is a smallest algebra $\mathcal{Q}$ which contains $\mathcal{C}$.

(b) Define outer measure of a subset of $\mathbb{R}$.
Prove : The outer measure of an interval $[a, b]$ is $b-a$.

(c) Define measurable set.
Prove : If $A \subset \mathbb{R}$ and if $E_1, E_2, \ldots\ldots E_n$ are disjoint measurable sets; then

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*\left(A \cap E_i\right)$$

(d) Define Borel set.
Prove : Every Borel set is measurable.

(e) Show that : The CANTOR set has measure zero.

(f) Show that : m^* is translation invariant.

2 Attempt any **three** : 18

(a) Define measurable function.
Prove : If f is a measurable function; then f^2 is also measurable.

(b) State and prove : Egoroff's Theorem.

(c) Prove : If f be a bounded function defined on $[a, b]$ and if f is Riemann integrable on $[a, b]$; then f is measurable and

$$R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$$

(d) State and prove : Bounded Convergence Theorem.

(e) Prove : If f be a bounded measurable function defined on a set E of finite measure and if $A \le f(x) \le B$; then $A \cdot m(E) \le \int_E f \le B \cdot m(E)$.

(f) Prove : A bounded function f on $[a, b]$ is Riemann integrable iff the set of points at which f is discontinuous has measure zero.

3 Attempt any **three** : 18

(a) State and prove : Fatou's Lemma.

(b) State and prove : Monotone Convergence Theorem.

(c) State and prove : Lebesgue Convergence Theorem.

(d) If f be a non-negative integrable function over a set E and if $\varepsilon > 0$ be given; then prove that there is a $\delta > 0$ s.t.

$\int_A f < \varepsilon$; whenever $A \subset E$ and $m(A) < \delta$.

(e) Prove : If f is integrable over E; then $|f|$ is also integrable over E and $\left| \int_E f \right| \leq \int_E |f|$.

4 Attempt any **three** : **18**

(a) If f is of bounded variation on $[a, b]$; then prove that :

$T_a^b = P_a^b + N_a^b$ and $f(b) - f(a) = P_a^b - N_a^b$

(b) If f is integrable on $[a, b]$ and $\int_a^x f(t)\,dt = 0;\ \forall x \in [a, b];$ then $f(t) = 0$ a.e. in $[a, b]$.

(c) If f be an integrable function on $[a, b]$ and if $F(x) = F(a) + \int_a^x f(t)\,dt;$ then prove that :

$F'(x) = f(x)$ for almost all $x \in [a, b]$.

(d) Prove : A function F is an indefinite integral iff it is absolutely continuous.

(e) State and prove : Vitali Theorem.

5 Attempt any **six** : 18

(a) Show that : If A is countable, then $m^*(A) = 0$.

(b) Show that : If E_1 and E_2 are measurable; then $m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$

(c) Write down Littlewood's three principle.

(d) Show that : The sum and product of two simple functions are also simple.

(e) If $y \in [0, 1)$ and $E \subset [0, 1)$ be a measurable set; then prove that $E \overset{\circ}{+} y$ is measurable.

(f) If $f(x) = \begin{cases} 0 ; & \text{whenever } x \notin Q \\ 1 ; & \text{whenever } x \in Q \end{cases}$; then show that

$$R\overline{\int_a^b} f(x)\,dx = b - a.$$

(g) Show that : $D^+[-f(x)] = -D_+f(x)$

(h) Show that :
If f is absolutely continuous, then f has a derivative almost everywhere.