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KA-473

Seat No. ________

B. Sc. (Sem. V) Examination

October / November – 2017

Mathematics : CC-MATH-501

(Group Theory)

Time : **3** Hours] [Total Marks : **70**

Instructions : (1) **All** questions are **compulsory**.

(2) The **figures** to the **right** indicate the marks of the corresponding questions.

1 (a) Define a group. **6**

Examine each set given below and determine whether it is a group under the binary operation $*$. If it is a group, then obtain its identity and if it is not a group then find out which postulates are not satisfied.

(i) Set $\mathbb{Z}$ with $a * b = a - b$

(ii) Set $\mathbb{N}$ with $a * b = a \cdot b$

(iii) Set $\left\{ z \in \mathbb{C} \;/\; |z| = 1 \right\}$ with $a * b = a \cdot b$

(b) State and prove Lagrange's theorem. **6**

(c) In a finite group, prove that each element is of a finite order. **6**

OR

1 (a) State and prove the necessary and sufficient conditions for a non-empty subset H of a group G to be a subgroup of G. **6**

(b) Prove that a group G is commutative if $(ab)^n = a^n b^n,\ a,\ b \in G$, for three consecutive integer n. 6

(c) If $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}_2 \right\}$. 6

Show that G is a commutative group under matrix addition. Also find the order of G.

2 (a) Prove that the order of a permutation $f \in S_n$ is the least common multiple of the length of its disjoint cycles. 6

(b) If $G = \left\{ e, a, a^2, a^3, \ldots\ldots, a^{19} \middle| a^{20} = e \right\}$ is a cyclic group of order 20 and $H = \left\langle a^4 \right\rangle$, then prepare the group table for the quotient group G/H. Using group table, answer the following 6

(i) Find the inverse of Ha^3 in G/H

(ii) Solve the equation $\left(Ha^3 \right) x = Ha^2$ in G/H

(c) Show that isomorphism between two groups is an equivalence relation. 6

OR

2 (a) Show that any two disjoint cycles in S_n are commutative. 6

(b) If for a subgroup H of a group G, the product of two right cosets of H in G is again a right coset of H in G then prove that H is a normal subgroup of G. 6

(c) In usual notations, if G is a group with $a(G) = \{I_G\}$ then show that **6**

(i) G is commutative

(ii) $a^2 = e$ for each $a \in G$.

3 (a) Show that, in a finite cyclic group, the group and its generator have the same order. **6**

(b) Show that a cyclic group of order n has exactly $\phi(n)$ generators. Where ϕ is the Euler's phi function. **6**

Moreover, what can you say about the generators of an infinite cyclic group ?

Justify your answer.

(c) Show that Kernel K_ϕ of a homomorphism $\phi : (G, 0) \to (G', *)$ is a normal subgroup of G. **6**

OR

3 (a) State and prove the first fundamental theorem of homomorphism. **6**

(b) Show that a cyclic group of order eight is homomorphic to a cyclic group of order four. **6**

(c) If $G \neq \{e\}$ is a group having no proper subgroup then show that G is a cyclic group of prime order. **6**

4 Attempt any **two** : 8

(a) Show that a group of order five is always commutative.

(b) Show that the set

$H = \left\{ f \in S_n \,\middle|\, 1 \text{ is invariant under } f \right\}$ is a subgroup of S_n.

(c) The set $G = R \sim \{-1\}$ is a group under binary operation $*$, where $a * b = a + b + ab$, $a, b \in G$ and if $G' = (R_0, \bullet)$, where R_0 is the set of all non-zero real numbers, then show that $G \cong G'$.

5 Attempt any **two** : 8

(a) Using the Euler's theorem, find the remainder obtained on dividing 3^{256} by 14.

(b) Prove that

(i) A subgroup of index 2 in a group is a normal subgroup.

(ii) The alternating subgroup A_n of symmetric group S_n is a normal subgroup of S_n for each $n \geq 2$.

(c) Prove that a homomorphism

$\phi : (G, 0) \rightarrow (G', *)$ is one-one if and only if $K_\phi = \{e\}$.
